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# PACKING DIRECTED CIRCUITS

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We prove a conjecture of Younger, that for every integer $n \geq 0$ there exists an integer $t \geq 0$ such that for every digraph $G$, either $G$ has $n$ vertex-disjoint directed circuits, or $G$ can be made acyclic by deleting at most $t$ vertices.

## 1. Introduction

A well-known theorem of Erdős and Pósa [1] states the following.

**(1.1)** *For any integer $n \geq 0$ there exists an integer $t \geq 0$ such that for every graph $G$, either $G$ has $n$ circuits that are pairwise vertex-disjoint, or there exists $T \subseteq V(G)$ with $|T| \leq t$ such that $T$ meets every circuit of $G$.*

(All graphs and digraphs in this paper are finite, and may have loops or multiple edges. We denote the vertex- and edge-sets of a graph or digraph $G$ by $V(G)$ and $E(G)$.)

In 1973, Younger [8] conjectured that there was an analogue of (1.1) for directed circuits in digraphs. (The question probably occurred to other workers before then — for instance, it seems that Fulkerson worked on it in the 1950's, but put nothing in print.) For $n=0$ and 1 this is obvious. For $n=2$ it was conjectured independently by

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Gallai [4] (who also considered the $n>2$ case without publishing it), and eventually was answered by McCuaig [5] in the following strong form.

**(1.2)** *For every digraph $G$, either $G$ has two vertex-disjoint directed circuits, or there exists $T \subseteq V(G)$ with $|T| \leq 3$ such that $T$ meets every directed circuit of $G$.*

But Younger's conjecture in general (and even for $n=3$) has remained open until the present. In this paper we prove the conjecture.

For a digraph $G$ we denote by $\nu(G)$ the maximum $n$ such that $G$ has $n$ directed circuits, pairwise vertex-disjoint, and by $\tau(G)$ the minimum $t$ such that there exists $T \subseteq V(G)$ with $|T| = t$ meeting all directed circuits. Evidently $\nu(G) \leq \tau(G)$, and Younger's conjecture is that $\tau(G)$ is at most $f(\nu(G))$ for some function $f$ independent of $G$. Thus, the main result of this paper is the following.

**(1.3)** *For every integer $n \geq 0$ there exists a (minimum) integer $t_n \geq 0$ such that for every digraph $G$, either $\nu(G) \geq n$ or $\tau(G) \leq t_n$.*

Throughout the paper, $t_n$ is defined as in (1.3); we shall prove by induction on $n$ that $t_n$ exists for all $n$. Obviously $t_0 = t_1 = 0$, and from McCuaig's theorem $t_2 = 3$.

We shall (implicitly) prove an upper bound on $t_n$, but it is very large (a multiply iterated exponential, where the number of iterations is also a multiply iterated exponential) and probably far from best possible. The best *lower* bound on $t_n$ that we know is $O(n\log(n))$, due to Alon (unpublished).

The paper is organized as follows. In section 2 we apply Ramsey's theorem to deduce a lemma with several consequences. One of them, not needed here but perhaps of some independent interest, is that for any $n \geq 0$ there exists $t \geq 0$ so that for every digraph $G$, either $G$ has a list of $n$ directed circuits with each vertex in at most two of them, or $\tau(G) \leq t$. (A weak form of this was proved in [7].) Also, the lemma implies that to prove (1.3) in general, it suffices to prove it for digraphs in which each vertex has indegree and outdegree 2, and with a number of other properties.

In section 3 we prove a second lemma that says roughly that if $G$ has as a subdigraph a kind of grid together with some additional paths, then $\nu(G)$ is large. Then in section 4, we prove that digraphs with the properties implied by the lemma of section 2 do contain such grids. Finally, in section 5 we use (1.3) to obtain a polynomial-time algorithm for the problem.

Let us establish some notation and terminology. A *path* (all paths in this paper are "directed paths") has at least one vertex, and no "repeated" vertices. If $a$ and $b$ are its first and last vertices we call it an *$(a,b)$-path* or a *path from $a$ to $b$*. If $x,y$ are vertices of a path $P$, and there is a subpath of $P$ from $x$ to $y$, we denote this subpath by $P[x,y]$. A *linkage* $L$ in a digraph $G$ is a subdigraph consisting of vertex-disjoint paths, called the *components* of $L$. If a linkage $L$ has components $P_1,\ldots,P_k$ and $P_i$ is an $(a_i,b_i)$-path $(1 \leq i \leq k)$ we say that $L$ *links* $(a_1,\ldots,a_k)$ to

$(b_1,\dots,b_k)$, and if $A,B \subseteq V(G)$ with $a_1,\dots,a_k \in A$ and $b_1,\dots,b_k \in B$ we say the linkage is *from A to B*. We call $k$ the *size* of $L$.

A *separation* in a digraph $G$ is a pair $(X,Y)$ of subsets of $V(G)$ with $X \cup Y = V(G)$, so that no edge of $G$ has tail in $X-Y$ and head in $Y-X$. Its *order* is $|X \cap Y|$. We shall frequently need the following version of Menger's theorem for digraphs:

**(1.4)** *Let $G$ be a digraph, let $k \geq 0$ be an integer, and let $A$, $B \subseteq V(G)$. Then exactly one of the following holds:*
(i) *there is a linkage from $A$ to $B$ of size $k$*
(ii) *there is a separation $(X,Y)$ of $G$ of order $<k$ with $A \subseteq X$ and $B \subseteq Y$.*

We shall also need the following result of Erdős and Szekeres [2].

**(1.5)** *Let $s,t \geq 1$ be integers, and let $n=(s-1)(t-1)+1$. Let $a_1,\dots,a_n$ be distinct integers. Then either*
(i) *there exist $1 \leq i_1 < \dots < i_s \leq n$ so that $a_{i_1} < a_{i_2} < \dots < a_{i_s}$, or*
(ii) *there exist $1 \leq i_1 < \dots < i_t \leq n$ so that $a_{i_1} > a_{i_2} > \dots > a_{i_t}$.*

The third standard combinatorial result we use is Ramsey's theorem [6], the following.

**(1.6)** *For all integers $q$, $l$, $r \geq 1$ there exists a (minimum) integer $R_l(r;q) \geq 0$ so that the following holds. Let $Z$ be a set with $|Z| \geq R_l(r;q)$, let $Q$ be a set with $|Q|=q$, and for each $X \subseteq Z$ with $|X|=l$ let $f(X) \in Q$. Then there exists $S \subseteq Z$ with $|S|=r$ and there exists $x \in Q$ so that $f(X)=x$ for all $X \subseteq S$ with $|X|=l$.*

## 2. An application of Ramsey's theorem

In this section we prove that (1.3) is true in general if it is true for digraphs with the property that for every vertex $v$, the outdegree and indegree of $v$ are equal and are at most 2. We begin with the following. If $G$ is a digraph and $X \subseteq V(G)$, the digraph obtained by deleting $X$ is denoted by $G \backslash X$.

**(2.1)** *Let $n \geq 1$ be an integer such that $t_{n-1}$ exists. Let $G$ be a digraph with $\nu(G) < n$, and let $T \subseteq V(G)$ with $|T| = \tau(G)$, meeting all directed circuits of $G$. Let $A$, $B \subseteq T$ be disjoint with $|A|=|B|=r$ say, where $r \geq 2t_{n-1}$. Then there is a linkage in $G$ from $A$ to $B$ of size $r$ with no vertex in $T-(A \cup B)$.*

**Proof.** Suppose not, and let $Z = T-(A \cup B)$. By (1.4) applied to $G \backslash Z$ there is a separation $(X,Y)$ of $G$ with $A \subseteq X, B \subseteq Y, Z \subseteq X \cap Y$ and with $|X \cap Y - Z| < r$. Let $W = X \cap Y - Z$; then $|W| < |B|$. Let $T_1 = A \cup (X \cap Y)$. Since

$$|T_1| \leq |A-Y| + |Z| + |W| < |A-Y| + |Z| + |B| \leq |T| = \tau(G)$$

there is a directed circuit $C_1$ of $G$ with $T_1 \cap V(C_1) = \emptyset$. Since $T \cap V(C_1) \neq \emptyset$ it follows that $B \cap V(C_1) \neq \emptyset$ and hence $Y \cap V(C_1) \neq \emptyset$. Since $(X,Y)$ is a separation and $X \cap Y \cap V(C_1) = \emptyset$, it follows that $V(C_1)$ does not meet $X$, and so $V(C_1) \subseteq Y - X$. Similarly there is a directed circuit $C_2$ with $V(C_2) \subseteq X - Y$.

Let $G_1 = G \backslash Y$. Since $\nu(G) < n$ and $V(C_1) \cap V(G_1) = \emptyset$, it follows that $\nu(G_1) < n-1$, and consequently $\tau(G_1) \leq t_{n-1}$. Similarly $\tau(G_2) \leq t_{n-1}$, where $G_2 = G \backslash X$. But every directed circuit of $G$ that is not a circuit of $G_1$ or $G_2$ meets $X \cap Y$, and so

$$\begin{aligned} \tau(G) &\leq \tau(G_1) + \tau(G_2) + |X \cap Y| < 2t_{n-1} + |Z| + r \\ &= 2t_{n-1} + |T| - r = 2t_{n-1} + \tau(G) - r \end{aligned}$$

and so $r < 2t_{n-1}$, contrary to hypothesis. The result follows. ∎

The main result of this section is the following. It was proved in joint work with Noga Alon.

**(2.2)** *Let $n \geq 1$ be an integer such that $t_{n-1}$ exists, and let $k \geq 1$ be an integer. Then there exists an integer $t \geq 0$ so that the following holds. Let $G$ be a digraph with $\nu(G) < n$ and $\tau(G) \geq t$. Then there are distinct vertices $a_1,\ldots,a_k$, $b_1,\ldots,b_k$ of $G$ and two linkages $L_1, L_2$ of $G$, both of size $k$, so that*

(i) *$L_1$ links $(a_1,\ldots,a_k)$ to $(b_1,\ldots,b_k)$*

(ii) *$L_2$ links $(b_1,\ldots,b_k)$ to one of $(a_1,\ldots,a_k),(a_k,\ldots,a_1)$*

(iii) *every directed circuit of $L_1 \cup L_2$ meets $\{a_1,\ldots,a_k,b_1,\ldots,b_k\}$.*

**Proof.** Let $l = (k-1)^2+1$, $r = \max\{2t_{n-1},(k+1)l\}$, and $q = (l!+1)^2$. Let $t = R_l(r;q)+l$, where $R_l(r;q)$ is as in (1.6). Then $r \geq l$ and $t \geq 2r$, since trivially $R_l(r;q) \geq 2r-1$. We claim that $t$ satisfies the theorem. For let $G$ be a digraph with $\nu(G) < n$ and $\tau(G) \geq t$. Choose $T \subseteq V(G)$ with $|T| = \tau(G)$, meeting all directed circuits of $G$. Choose $A \subseteq T$ with $|A| = l$, and let $Z = T - A$. Thus, $|Z| \geq R_l(r;q)$. Let $Z = \{z_i : 1 \leq i \leq |Z|\}$.

For each $X \subseteq Z$ with $|X| = x$ say, let $X = \{z_{i_1},\ldots,z_{i_x}\}$ where $i_1 < \ldots < i_x$; we denote the $x$-tuple $(z_{i_1},\ldots,z_{i_x})$ by $\bar{X}$, and for $1 \leq h \leq x$ we denote $z_{i_h}$ by $\bar{X}(h)$.

Let $X \subseteq Z$ with $|X| = l$. If there exists a linkage $L_1(X)$ say in $G$ from $A$ to $X$ with no vertex in $Z - X$, then there is a sequence $(a_1,a_2,\ldots,a_l)$ with $\{a_1,a_2,\ldots,a_l\} = A$ so that $L_1(X)$ links $(a_1,a_2,\ldots,a_l)$ to $\bar{X}$, and we define $p_1(X)$ to be $(a_1,a_2,\ldots,a_l)$; if no such linkage exists we define $p_1(X)$ to be $\emptyset$. Similarly, if there exists a linkage $L_2(X)$ say in $G$ from $X$ to $A$ with no vertex in $Z - X$, we define $p_2(X)$ to be $(b_1,b_2,\ldots,b_l)$, where $\{b_1,b_2,\ldots,b_l\} = A$ and $L_2(X)$ links $\bar{X}$ to $(b_1,b_2,\ldots,b_l)$; if no such linkage exists we define $p_2(X)$ to be $\emptyset$. We define $f(X)$ to be the pair $(p_1(X),p_2(X))$. Let $Q$ be the set of all pairs $(a,b)$ such that each of $a,b$ is either the empty set, or a sequence $(x_1,x_2,\ldots,x_l)$ with $\{x_1,x_2,\ldots,x_l\} = A$. Then $|Q| = (l!+1)^2$, and $f(X) \in Q$ for each $X \subseteq Z$ with $|X| = l$. By (1.6), there exist

$S \subseteq Z$ with $|S|=r$, and $(a,b) \in Q$, so that $f(X)=(a,b)$ for all $X \subseteq S$ with $|X|=l$. We claim that $a \neq \emptyset$ and $b \neq \emptyset$. Indeed, suppose for a contradiction that $a=\emptyset$, and choose a set $A'$ with $A \subseteq A' \subseteq T-S$ and $|A'|=r$. By (2.1) there is a linkage $L'$ in $G$ from $A'$ to $S$ of size $r$ with no vertex in $T-(A' \cup S)$. The linkage $L'$ includes a linkage from $A$ to some $X \subseteq S$ with no vertex in $T-(A \cup X)$. Thus $a=p_1(X) \neq \emptyset$, and similarly $b \neq \emptyset$. This proves the claim.

Let $a=(a_1,a_2,\ldots,a_l)$ and $b=(b_1,b_2,\ldots,b_l)$. Consequently, for all $X \subseteq S$ with $|X|=l$, $L_1(X)$ links $(a_1,\ldots,a_l)$ to $\bar{X}$ and $L_2(X)$ links $\bar{X}$ to $(b_1,\ldots,b_l)$.

For $1 \leq i \leq l$, choose $j$ with $1 \leq j \leq l$ so that $a_i = b_j$, and define $j_i = j$. Then $j_1,\ldots,j_l$ are distinct integers. By (1.5), since $l = (k-1)^2+1$, there exist $1 \leq i_1 < i_2 < \ldots < i_k \leq l$ so that either

$$j_{i_1} < j_{i_2} < \ldots < j_{i_k}$$

or

$$j_{i_1} > j_{i_2} > \ldots > j_{i_k}.$$

Define $(i'_1,\ldots,i'_k)$ to be $(j_{i_1},\ldots,j_{i_k})$ in the first case, and $(j_{i_k},\ldots,j_{i_1})$ in the second. Hence $i'_1 < \ldots < i'_k$.

Let $D=\{\bar{S}(l),\bar{S}(2l),\ldots,\bar{S}(kl)\}$. Choose $X \subseteq S$ with $|X|=l$ so that $\bar{S}(hl) = \bar{X}(i_h)$ for $1 \leq h \leq k$. (This is possible since $i_1 < \ldots < i_k$ and there are $\geq l-1$ terms in $\bar{S}$ between any two selected to be in $D$.) The linkage $L_1(X)$ links $(a_1,\ldots,a_l)$ to $\bar{X}$, and consequently it includes a linkage $L_1$ linking $(a_{i_1},\ldots,a_{i_k})$ to $\bar{D}$. Moreover, the only vertices of $T$ in $L_1$ belong to $A \cup D$.

Similarly choose $Y \subseteq S$ with $|Y|=l$ so that for $1 \leq h \leq k$, $\bar{S}(hl)=\bar{Y}(i'_h)$. Since $L_2(Y)$ links $\bar{Y}$ to $(b_1,\ldots,b_l)$, it includes a linkage $L_2$ linking $\bar{D}$ to $(b_{i'_1},\ldots,b_{i'_k})$. Now $(b_{i'_1},\ldots,b_{i'_k})$ is either $(b_{j_{i_1}},\ldots,b_{j_{i_k}}) = (a_{i_1},\ldots,a_{i_k})$ or its reverse. Moreover, every directed circuit in $L_1 \cup L_2$ meets $T$, and the only vertices of $T$ in $V(L_1 \cup L_2)$ are $a_{i_1},\ldots,a_{i_k}$ and $\bar{S}(l),\ldots,\bar{S}(kl)$; and so $L_1, L_2$ satisfy the theorem. ∎

Theorem (2.2) is quite powerful. For instance, define $\nu_2(T)$ to be the maximum number of directed circuits of $G$ (with repetition allowed) so that every vertex is in at most two of them. Evidently

$$2\nu(G) \leq \nu_2(G) \leq 2\tau(G)$$

and since we are trying to bound $\tau(G)$ above by a function of $\nu(G)$, it should be easier to bound it above by a function of $\nu_2(G)$. This is indeed the case, and such a bound follows from (2.2), because the two linkages $L_1$ and $L_2$ imply that $\nu_2(G) \geq \frac{1}{2}k$. (Make a circuit from the $i$th and $(k+1-i)$th component of each linkage, for $1 \leq i \leq \frac{1}{2}k$.) Of course, (2.2) assumes that $t_{n-1}$ exists, but that can be replaced by the hypothesis that if $\nu_2(G) < n-1$ then $\tau(G)$ is bounded above by a function of $\nu_2(G)$, modifying (2.1) accordingly. We omit the details since we are going to prove a stronger result.

We say a digraph is *divalent* if every vertex has indegree 2 and outdegree 2, or indegree 1 and outdegree 1. In sections 3 and 4 we shall prove the following:

**(2.3)** *For every integer $n \geq 1$ there exists an integer $k \geq 0$ such that for every divalent digraph $G$, if there are two linkages $L_1, L_2$ in $G$, both of size $\geq k$, so that each component of $L_1$ meets each component of $L_2$ and $L_1 \cup L_2$ has no directed circuits, then $\nu(G) \geq n$.*

We shall assume (2.3) for the moment; in the remainder of this section we show how to combine (2.2) and (2.3) to prove the main result (1.3). First, we show:

**(2.4)** *For every integer $n \geq 1$ there exists $k \geq 0$ so that the following holds. Let $G$ be a digraph, and let $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ be distinct vertices of $G$. Let $L_1, L_2$ be linkages in $G$ linking $(a_1, \ldots, a_k)$ to $(b_1, \ldots, b_k)$, and $(b_1, \ldots, b_k)$ to one of $(a_1, \ldots, a_k)$, $(a_k, \ldots, a_1)$ respectively. Let every directed circuit of $L_1 \cup L_2$ meet $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$. Then $\nu(G) \geq n$.*

**Proof, assuming (2.3).** Given $n \geq 1$, choose $k' \geq 0$ so that $n$, $k'$ satisfy (2.3) (with $k$ replaced by $k'$); we may assume that $6k' \geq n$, by increasing $k'$. Let $k = 2R_2(6k';9)$, defined as in (1.6). We claim that $n$ and $k$ satisfy (2.4). For let $G, a_1, \ldots, a_k, b_1, \ldots, b_k$, $L_1, L_2$ be as in (2.4). We show by induction on $|E(G)| + |V(G)|$ that $\nu(G) \geq n$. If $L_1 \cup L_2 \neq G$ the result follows immediately by induction, so we may assume that $L_1 \cup L_2 = G$. If some edge $e$ belongs to $L_1$ and to $L_2$, the result follows from the inductive hypothesis by contracting $e$. (For let this operation result in a digraph $G'$. Since $e$ belongs to a component of $L_1$ and to a component of $L_2$, and since $L_1 \cup L_2 = G$, the head of $e$ is the head of no other edge of $G$, and so $\nu(G') = \nu(G)$. Moreover, there are clearly linkages in $G'$ satisfying the hypotheses of (2.4).)

We therefore may assume that every edge of $G$ belongs to exactly one of $L_1, L_2$. In particular, $G$ is divalent.

For $1 \leq i \leq k$, let $P_i$ be the component of $L_1$ with first vertex $a_i$, and let $Q_i$ be the component of $L_2$ with first vertex $b_i$.

**(1)** *If there exist $I$, $J \subseteq \{1, \ldots, k\}$ with $|I| = |J| = 3k'$ so that $P_i$ meets $Q_j$ for all $i \in I$ and $j \in J$, then $\nu(G) \geq n$.*

**Subproof.** Choose $I' \subseteq I$ with $|I'| = k'$. There are $2k'$ vertices that are ends of paths $P_i$ $(i \in I')$, and each of them is an end-vertex of at most one $Q_j$ $(j \in J)$. Since $|J| = 3k'$ there exists $J' \subseteq J$ with $|J'| = k'$ so that $P_i$ and $Q_j$ have no common end-vertex for $i \in I'$ and $j \in J'$. Let $L'_1 \subseteq L_1$ be the union of the components $P_i$ $(i \in I')$, and define $L'_2 \subseteq L_2$ similarly. Now every directed circuit in $L_1 \cup L_2$ meets $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$, and each of $a_1, \ldots, a_k, b_1, \ldots, b_k$ is incident with at most one edge of $L_1' \cup L_2'$ (since $P_i$ and $Q_j$ have no common end-vertex for $i \in I'$ and $j \in J'$) and so $L_1' \cup L_2'$ has no directed circuits. From (2.3), $\nu(G) \geq n$. This proves (1).

For $1 \leq h < i \leq \frac{1}{2}k$, define $f(\{h,i\})$ as follows. If $P_i \cup P_{k+1-i}$ is disjoint from $Q_h \cup Q_{k+1-h}$ and $Q_i \cup Q_{k+1-i}$ is disjoint from $P_h \cup P_{k+1-h}$, let $f(\{h,i\}) = 0$. Otherwise, at least one of the eight digraphs

$$P_i \cap Q_h$$
$$P_i \cap Q_{k+1-h}$$
$$P_{k+1-i} \cap Q_h$$
$$P_{k+1-i} \cap Q_{k+1-h}$$
$$Q_i \cap P_h$$
$$Q_i \cap P_{k+1-h}$$
$$Q_{k+1-i} \cap P_h$$
$$Q_{k+1-i} \cap P_{k+1-h}$$

is non-null. Number them $1,\ldots,8$ in order; we define $f(\{h,i\})$ to be the number of the first which is non-null. Thus, for all $h,i$ with $1 \leq h < i \leq \frac{1}{2}k$, $f(\{h,i\}) \in \{0,1,\ldots,8\}$.

Since $k = 2R_2(6k';9)$ there exists $S \subseteq \{1,\ldots,\frac{1}{2}k\}$ with $|S| = 6k'$ and $x$ with $0 \leq x \leq 8$ such that $f(\{h,i\}) = x$ for all $h,i \in S$ with $h < i$. Let $S = I \cup J$, where $I$ and $J$ both have cardinality $3k'$ and $i < j$ for all $i \in I$ and $j \in J$. If $x \neq 0$ then $\nu(G) \geq n$ by (1) (using $I$ and $J$). If $x = 0$, then the subdigraphs

$$G_i = P_i \cup P_{k+1-i} \cup Q_i \cup Q_{k+1-i}$$

are pairwise vertex-disjoint for all $i \in S$. But $P_i, P_{k+1-i}$ are $(a_i,b_i)$- and $(a_{k+1-i},b_{k+1-i})$-paths, and either $Q_i, Q_{k+1-i}$ are $(b_i,a_i)$- and $(b_{k+1-i},a_{k+1-i})$-paths, or they are $(b_i,a_{k+1-i})$- and $(b_{k+1-i},a_i)$-paths. In either case each $G_i$ has a directed circuit, and so

$$\nu(G) \geq |S| = 6k' \geq n$$

as required. ∎

**Proof of (1.3), assuming (2.3).** We prove (1.3) by induction on $n$; we therefore assume that $n \geq 1$ and $t_{n-1}$ exists, and we show that $t_n$ exists. Let $k$ be as in (2.4), and let $t$ be as in (2.2). We claim that there is no digraph $G$ with $\nu(G) < n$ and $\tau(G) \geq t$. For suppose that $G$ is such a digraph. By (2.2), there exist $a_1,\ldots,a_k,b_1,\ldots,b_k$ and $L_1, L_2$ as in (2.2), and so $\nu(G) \geq n$ by (2.4), a contradiction. Thus there is no such $G$, and consequently $t_n$ exists and $t_n < t$. ∎

## 3. Using a grid

In this section we show that, if a digraph $G$ contains a kind of grid, with some additional paths, then $\nu(G)$ is large. This is a lemma that will be used in section 4 to prove (2.3).

Let $p,q \geq 1$ be integers. A *$(p,q)$-fence* in a digraph $G$ is a sequence $(P_1,\ldots,P_{2p},Q_1,\ldots,Q_q)$ with the following properties:

(i) $P_1,\ldots,P_{2p}$ are mutually disjoint paths of $G$, and so are $Q_1,\ldots,Q_q$

(ii) for $1 \leq i \leq 2p$ and $1 \leq j \leq q, P_i \cap Q_j$ is a path (and therefore non-null)

(iii) for $1 \leq j \leq q$, the paths $P_1 \cap Q_j,\ldots,P_{2p} \cap Q_j$ are in order in $Q_j$, and the first vertex of $Q_j$ is in $V(P_1)$ and its last vertex is in $V(P_{2p})$

(iv) for $1 \leq i \leq 2p$, if $i$ is odd then $P_i \cap Q_1,\ldots,P_i \cap Q_q$ are in order in $P_i$, and if $i$ is even then $P_i \cap Q_q,\ldots,P_i \cap Q_1$ are in order in $P_i$.

Thus, each $(p,q)$-fence is a planar digraph with no directed circuits. Let $Q_j$ be an $(a_j,b_j)$-path $(1 \leq j \leq q)$; we call $\{a_1,\ldots,a_q\}$ the *top* of the fence, and $\{b_1,\ldots,b_q\}$ its *bottom*.

The main result of this section is the following.

**(3.1)** *For every integer $n \geq 1$, there are integers $p$, $r \geq 1$ with the following property. For any $q \geq 1$, let $(P_1,\ldots,P_{2p},Q_1,\ldots,Q_q)$ be a $(p,q)$-fence in a digraph $G$, and let there be $r$ disjoint paths in $G$ from the bottom of the fence to the top. Then $\nu(G) \geq n$.*

To prove (3.1) we need some preliminary lemmas. The first is:

**(3.2)** *Let $(P_1,\ldots,P_{2p},Q_1,\ldots,Q_q)$ be a $(p,q)$-fence in a digraph $G$, with top $A$ and bottom $B$. Let $A' \subseteq A$ and $B' \subseteq B$ with $|A'|=|B'|=r$ say, where $r \leq p$. Then there are paths $Q'_1,\ldots,Q'_r$ in $P_1 \cup \ldots \cup P_{2p} \cup Q_1 \cup \ldots \cup Q_q$ so that $(P_1,\ldots,P_{2p},Q'_1,\ldots,Q'_r)$ is a $(p,r)$-fence with top $A'$ and bottom $B'$.*

**Proof.** Let $G' = P_1 \cup \ldots \cup P_{2p} \cup Q_1 \cup \ldots \cup Q_q$. For $1 \leq j \leq q$, let $Q_j$ be an $(a_j,b_j)$-path. Suppose that $(X,Y)$ is a separation of $G'$ of order $< r$ with $A' \subseteq X$ and $B' \subseteq Y$. Since $|A'|=r$ there exists $j_1$ with $1 \leq j_1 \leq q$ so that $a_{j_1} \in A'$ and $V(Q_{j_1}) \cap X \cap Y = \emptyset$. Similarly there exists $j_2$ with $1 \leq j_2 \leq q$ so that $b_{j_2} \in B'$ and $V(Q_{j_2}) \cap X \cap Y = \emptyset$. Without loss of generality (by reversing the direction of all edges and considering the fence $(P_{2p},\ldots,P_1,Q_q,\ldots,Q_1)$) we may assume that $j_1 \leq j_2$. Since $p \geq r$, there exists $i$ with $1 \leq i \leq p$ such that $V(P_{2i-1}) \cap X \cap Y = \emptyset$. But $Q_{j_1} \cup P_{2i-1} \cup Q_{j_2}$ contains a path from $A'$ to $B'$ with no vertex in $X \cap Y$, a contradiction.

Hence there is no such $(X,Y)$. By (1.4) there are $r$ disjoint paths of $G'$ from $A'$ to $B'$, and then the result follows easily from the fact that $G'$ is planar and has no directed circuits. ∎

From (3.2) we deduce:

**(3.3)** *Let $n \geq 1$ be an integer, and let $p \geq 2n$ and $N \geq 2n^2 - 3n + 2$ be integers. For some integer $q \geq 1$ let $(P_1, \ldots, P_{2p}, Q_1, \ldots, Q_q)$ be a $(p,q)$-fence in a digraph $G$. Let $R_1, \ldots, R_N$ be disjoint paths of $G$ from the bottom of the fence to the top, so that each $R_k$ has no vertex or edge in*

$$P_1 \cup \ldots \cup P_{2p} \cup Q_1 \cup \ldots \cup Q_q$$

*except its end-vertices. Then $\nu(G) \geq n$.*

**Proof.** For $1 \leq j \leq q$, let $Q_j$ be an $(a_j, b_j)$-path. For $1 \leq k \leq N$, let $a_{s(k)}$ be the last vertex of $R_k$, and let $b_{t(k)}$ be the first vertex of $R_k$. By renumbering $R_1, \ldots, R_N$ we may assume that $s(1) < s(2) < \ldots < s(N)$. Let

$$G' = P_1 \cup \ldots \cup P_{2p} \cup Q_1 \cup \ldots \cup Q_q.$$

By (1.5), either there exist $1 \leq k_1 < k_2 < \ldots < k_n \leq N$ so that

$$t(k_1) < t(k_2) < \ldots < t(k_n)$$

or there exist $1 \leq k_1 < k_2 < \ldots < k_{2n} \leq N$ so that

$$t(k_1) > t(k_2) > \ldots > t(k_{2n}).$$

In the first case, by (3.2) there exist $Q'_1, \ldots, Q'_n \subseteq G'$ so that $(P_1, \ldots, P_{2p}, Q'_1, \ldots, Q'_n)$ is a $(p,n)$-fence with top $\{a_{s(k_1)}, \ldots, a_{s(k_n)}\}$ and bottom $\{b_{t(k_1)}, \ldots, b_{t(k_n)}\}$; but then for $1 \leq j \leq n$, $Q'_j$ is an $(a_{s(k_j)}, b_{t(k_j)})$-path and so $Q'_1 \cup R_{k_1}, \ldots, Q'_n \cup R_{k_n}$ are $n$ vertex-disjoint circuits, and $\nu(G) \geq n$ as required. In the second case, by (3.2) there exist $Q'_1, \ldots, Q'_{2n} \subseteq G'$ so that $(P_1, \ldots, P_{2p}, Q'_1, \ldots, Q'_{2n})$ is a $(p, 2n)$-fence with top $\{a_{s(k_1)}, \ldots, a_{s(k_{2n})}\}$ and bottom $\{b_{t(k_1)}, \ldots, b_{t(k_{2n})}\}$. Since $k_1 > k_2 > \ldots > k_{2n}$ it follows that for $1 \leq j \leq 2n, Q'_j$ is an $(a_{s(k_j)}, b_{t(k_{2n+1-j})})$-path, and therefore

$$Q'_j \cup R_{k_{2n+1-j}} \cup Q'_{2n+1-j} \cup R_{k_j} \qquad (1 \leq j \leq n)$$

are $n$ vertex-disjoint directed circuits of $G$, and again $\nu(G) \geq n$. ∎

**Proof of (3.1).** We proceed by induction on $n$. If $n = 1$ the result is easy (taking $p = r = 1$) and so we may assume that $n \geq 2$. Let $n'$ be the least integer with $2n' \geq n$; then $1 \leq n' < n$, and so there exist $p', r' \geq 1$ so that the theorem holds (with $n, p, r$ replaced by $n', p', r'$). Let

$$N = 2n^2 - 3n + 2$$
$$p'' = p' + 2N + 1$$
$$p = 16n(r' + N) + 2n + 2p''$$
$$r = 8(r' + N).$$

We claim that $n$, $p$, $r$ satisfy the theorem.

For let $(P_1,\dots,P_{2p},Q_1,\dots Q_q)$ be a $(p,q)$-fence in a digraph $G$, with top $A$ and bottom $B$. Let $R_1,\dots,R_r$ be disjoint paths in $G$ from $B$ to $A$. From (3.2) we may assume that $q=r$. We may also assume (by replacing $P_{2i-1}$ by a subpath) that for $1\leq i\leq p$, $P_{2i-1}$ has first vertex in $V(Q_1)$ and last vertex in $V(Q_r)$, and similarly $P_{2i}$ has first vertex in $V(Q_r)$ and last vertex in $V(Q_1)$. For $1\leq j\leq r$ let $Q_j$ be an $(a_j,b_j)$-path.

For $1\leq j\leq r$, the paths $P_1\cap Q_j, P_2\cap Q_j,\dots,P_{2p}\cap Q_j$ are disjoint, and in order in $Q_j$. Hence there are $p$ disjoint subpaths $Q_{1j},\dots,Q_{pj}$ of $Q_j$ so that

(i) $V(Q_{1j})\cup\dots\cup V(Q_{pj})=V(Q_j)$, and

(ii) for $1\leq i\leq p$, $(P_{2i-1}\cup P_{2i})\cap Q_j\subseteq Q_{ij}$.

Choose such paths $Q_{1j},\dots,Q_{pj}$ for each value of $j$. For $1\leq i\leq p$ we define

$$D_i = P_{2i-1}\cup P_{2i}\cup\bigcup(Q_{ij} : 1\leq j\leq r).$$

Then $D_1,\dots,D_p$ are mutually disjoint, and

$$V(D_1\cup\dots\cup D_p) = V(P_1\cup\dots\cup P_{2p}\cup Q_1\cup\dots\cup Q_r).$$

**(1)** *There is a linkage $L_1$ in $G$ of size $\frac{1}{2}r$, so that*

*(i) each component of $L_1$ has first vertex in $B$ and last vertex in $V(D_i)$ for some $i$ with $1\leq i\leq p-p''$, and*

*(ii) for $1\leq i\leq p-p''$, at most one component of $L_1$ has last vertex in $V(D_i)$.*

**Subproof.** Let $G_0$ be obtained from $G$ by adding, for $1\leq i\leq p-p''$, a new vertex $y_i$ and an edge from $v$ to $y_i$ for each $v\in V(D_i)$.

Suppose there is a separation $(X,Y)$ of $G_0$ of order $<\frac{1}{2}r$, with $B\subseteq X$ and $y_1,\dots,y_{p-p''}\in Y$. Since $p-p''\geq\frac{1}{2}r$, there exists $i$ with $1\leq i\leq p-p''$ such that $y_i\notin X\cap Y$, and hence $y_i\in Y-X$. Consequently $V(D_i)\subseteq Y$, and so $Q_1,\dots,Q_r$ each have a vertex in $Y$. For $1\leq j\leq r$ let $R_j$ have last vertex $a_{s(j)}$. Since each vertex in $X\cap Y$ belongs to at most one of $Q_1,\dots,Q_r$ and at most one of $R_1,\dots,R_r$, and since $2|X\cap Y|<r$, there exists $j$ with $1\leq j\leq r$ such that $R_j\cup Q_{s(j)}$ has no vertex in $X\cap Y$. But $R_j\cup Q_{s(j)}$ contains a path from the first vertex of $R_j$ (which is in $B$ and hence in $X$) to each vertex of $Q_{s(j)}$; and since some vertex of $Q_{s(j)}$ is in $Y$, this is impossible. Thus there is no such separation $(X,Y)$. By (1.4) there are $\frac{1}{2}r$ disjoint paths in $G_0$ from $B$ to $\{y_1,\dots,y_{p-p''}\}$. This proves (1).

**(2)** *There exists $L_1$ as in (1) so that $V(L_1\cap D_i)\neq\emptyset$ for only $\frac{1}{2}r$ values of $i$ with $1\leq i\leq p-p''$.*

**Subproof.** Choose a minimal linkage $L_1$ satisfying (1). If $1\leq i\leq p-p''$ and $L_1\cap D_i$ is non-null, let $P$ be a component of $L_1$ that meets $D_i$. If no component of $L_1$ has last vertex in $V(D_i)$ we can replace $P$ by a subpath from $B$ to $V(D_i)$, contradicting the minimality of $L_1$. Thus some component of $L_1$ has last vertex in $V(D_i)$. Hence there are at most $\frac{1}{2}r$ such values of $i$ since $L_1$ has size $\frac{1}{2}r$. This proves (2).

By symmetry, we deduce

**(3)** *There is a linkage $L_2$ in $G$ of size $\frac{1}{2}r$, so that*

(i) *each component of $L_2$ has last vertex in $A$ and first vertex in $V(D_i)$ for some $i$ with $p''+1 \le i \le p$)*

(ii) *for $p''+1 \le i \le p$, at most one component of $L_2$ has first vertex in $V(D_i)$*

(iii) *there are only $\frac{1}{2}r$ values of $i$ with $p''+1 \le i \le p$ such that $V(L_2 \cap D_i) \neq \emptyset$.*

There are $p-2p''$ values of $i$ with $p''+1 \le i \le p-p''$, and $D_i \cap (L_1 \cup L_2)$ is non-null for $\le r$ of them. Since $p-2p'' \ge 2n(r+1)$, there exists $i_1$ with $p''+1 \le i_1 \le p-p''-2n+1$ so that $D_{i_1}, D_{i_1+1}, \ldots, D_{i_1+2n-1}$ are all disjoint from $L_1 \cup L_2$. Let $i_2 = i_1 + 2n - 1$.

For $1 \le j \le r$ let $a'_j$ be the first vertex of $Q_j$ in $P_{2i_1-1}$, let $b'_j$ be the last vertex of $Q_j$ in $P_{2i_2}$, and let $Q'_j = Q_j[a'_j, b'_j]$. Let $A' = \{a'_1, \ldots, a'_r\}$ and $B' = \{b'_1, \ldots, b'_r\}$. Let

$$G' = P_1 \cup \ldots \cup P_{2p} \cup Q_1 \cup \ldots \cup Q_r \cup L_1 \cup L_2,$$

$$G_2 = P_{2i_1-1} \cup P_{2i_1} \cup \ldots \cup P_{2i_2} \cup Q'_1 \cup \ldots \cup Q'_r.$$

**(4)** *If in $G'$ there are $N$ disjoint paths from $B'$ to $A'$ then $\nu(G) \ge n$.*

**Subproof.** Let $R$ be any minimal path in $G'$ from $B'$ to $A'$. We claim that no vertices or edges of $R$ belong to $G_2$ except the end-vertices of $R$. For certainly $R \not\subseteq G_2$ since there is no path of $G_2$ from $V(P_{2i_2})$ to $V(P_{2i_1-1})$. But every edge $e$ of $G'$ not in $E(G_2)$ but with tail or head in $V(G_2)$ belongs to some $E(Q_j)$, since $G_2 \cap (L_1 \cup L_2)$ is null and each $P_i$ is contained in or disjoint from $G_2$; and consequently every such edge has tail in $B'$ or head in $A'$. Since no internal vertex of $R$ is in $A' \cup B'$ from the minimality of $R$ it follows that the only such edges in $R$ are the first and the last; and this proves our claim that $R$ is disjoint from $G_2$ except for its end-vertices.

Now suppose that there are $N$ disjoint paths of $G'$ from $B'$ to $A'$. We may assume they are all minimal, and therefore each has no vertex or edge in $G_2$ except its end-vertices. From (3.3) applied to the $(2n,r)$-fence

$$(P_{2i_1-1}, P_{2i_1}, \ldots, P_{2i_2}, Q'_1, \ldots, Q'_r)$$

(which has top $A'$ and bottom $B'$) we deduce that $\nu(G) \ge n$. This proves (4).

From (4) and (1.4) we may assume that there is a separation $(X,Y)$ of $G'$ of order $< N$ with $B' \subseteq X$ and $A' \subseteq Y$. Let $H = G' \backslash Y$. We shall show that $\nu(H) \ge n'$.

Now $L_1$ has $\frac{1}{2}r$ components, each with first vertex in $B = \{b_1, \ldots, b_r\}$; let these components be $S_j$ $(j \in J)$ where $J \subseteq \{1, \ldots, r\}$, $|J| = \frac{1}{2}r$, and for $j \in J$, $S_j$ has first vertex $b_j$. For $j \in J$, let the last vertex of $S_j$ belong to $V(D_{f(j)})$. Thus, $1 \le f(j) \le p - p''$. For $j \in J$ we define $H_j$ to be the subdigraph of $G'$ induced on the vertex set

$$V(Q_j \cup S_j \cup D_{f(j)} \cup D_{f(j)+1}).$$

**(5)** *For each $v \in V(G)$ there are at most four values of $j \in J$ so that $v \in V(H_j)$.*

**Subproof.** Any vertex of $G$ belongs to at most one of $Q_1, \ldots, Q_r$, at most one of $S_j$ ($j \in J$), at most one of $D_{f(j)}$ ($j \in J$), and at one most of $D_{f(j)+1}$ ($j \in J$). Hence it belongs to $H_j$ for at most four values of $j$. This proves (5).

Let $I_0$ be the set of all $i$ with

$$p - p' - 2N + 1 \le i \le p$$

such that $V(D_{i-1} \cup D_i) \cap X \cap Y = \emptyset$. Since there are $p' + 2N$ possibilities for $i$, and $D_i$ meets $X \cap Y$ for at most $N$ of them, and $D_{i-1}$ meets $X \cap Y$ for at most $N$ of them, it follows that $|I_0| \ge p'$. Choose $I \subseteq I_0$ with $|I| = p'$.

**(6)** *$D_i \subseteq H$ for each $i \in I$.*

**Subproof.** Since $r > |X \cap Y|$, there exists $j$ with $1 \le j \le r$ so that $Q_j$ does not meet $X \cap Y$. Since $b'_j \in B' \subseteq X$ it follows that $Q_j[b'_j, b_j] \subseteq H$. Let $i \in I$. Since $b'_j$ belongs to $P_{2i_2}$, and

$$i_2 = i_1 + 2n - 1 \le p - p'' = p - p' - 2N - 1 \le i - 2,$$

it follows that $Q_j[b'_j, b_j]$ meets $P_{2i-3}$. Hence there exists $u \in X \cap V(P_{2i-3})$. Since $V(D_{i-1} \cup D_i)$ does not meet $X \cap Y$, every path of $D_{i-1} \cup D_i$ with first vertex $u$ is a path of $H$. Since $D_i$ is contained in the union of all such paths it follows that $D_i \subseteq H$. This proves (6).

Let $i_3$ and $i_4$ be the minimum and maximum members of $I$. For $1 \le j \le r$ let $a''_j$ be the first vertex of $Q_j$ in $P_{2i_3-1}$, and let $b''_j$ be the last vertex of $Q_j$ in $P_{2i_4}$.

**(7)** *For each $j \in J$ there is a path $T_j \subseteq H_j$ from $b''_j$ to $a''_j$.*

**Subproof.** Certainly there is a path from $b''_j$ to $b_j$, namely $Q_j[b''_j, b_j]$, and one from $b_j$ to some vertex $u$ of $D_{f(j)}$, namely $S_j$. Let $v$ be a vertex of $Q_j$ in $D_{f(j)+1}$; then there is a path of $G'$ with vertex set in $V(D_{f(j)} \cup D_{f(j)+1})$ from $u$ to $v$. We claim there is a path in $Q_j$ from $v$ to $a''_j$. To show this, we observe that

$$f(j) + 1 \le p - p'' + 1 = p - p' - 2N < i_3,$$

and so $v$ occurs in $Q_j$ before $a''_j$. This proves (7).

By (5), $H_j$ meets $X \cap Y$ for at most $4|X \cap Y| < 4N$ values of $j \in J$, and so there exists $J' \subseteq J$ with $|J'| \geq \frac{1}{2}r - 4N$ so that $V(H_j) \cap X \cap Y = \emptyset$ for all $j \in J'$. Since by (6), $T_j$ has first vertex in $X$ and does not meet $X \cap Y$, it follows that $T_j \subseteq H$ for all $j \in J'$. Let

$$A'' = \{a''_j : j \in J'\}$$
$$B'' = \{b''_j : j \in J'\}.$$

(8) *There are at least $r'$ disjoint paths in $H$ from $B''$ to $A''$.*

**Subproof.** Let $(X', Y')$ be a separation of $H$ with $B'' \subseteq X'$ and $A'' \subseteq Y'$. Since for all $j \in J'$, $T_j \subseteq H$ and has first vertex in $X'$ and last vertex in $Y'$, it follows that $T_j$ meets $X' \cap Y'$. By (5),

$$|X' \cap Y'| \geq \frac{1}{4}|J'| \geq \frac{1}{8}r - N \geq r'.$$

Hence (1.4) implies (8).

Now the paths $P_{2i-1}$ and $P_{2i}$ ($i \in I$), and $Q_j[a''_j, b''_j]$ ($j \in J'$) define a $(p', |J'|)$-fence in $H$, by (6) and the definition of $J'$; and this fence has top $A''$ and bottom $B''$. Hence from (8) and the choice of $p', r'$, it follows that $\nu(H) \geq n'$.

Thus, $\nu(G \backslash Y) \geq n'$. Similarly, $\nu(G \backslash X) \geq n'$, and so $\nu(G) \geq 2n' \geq n$, as required. ∎

## 4. Finding the grid

In this section we complete the proof of (1.3), by deriving (2.3) from (3.1).

Let $p$, $q \geq 1$ be integers. A *$(p,q)$-web* in a digraph $G$ is a sequence

$$(P_1, \ldots, P_p, Q_1, \ldots, Q_q)$$

such that

(i) $P_1, \ldots, P_p$ are mutually disjoint paths of $G$, and so are $Q_1, \ldots, Q_q$,
(ii) for $1 \leq i \leq p$ and $1 \leq j \leq q$, $P_i$ meets $Q_j$, and
(iii) $P_1 \cup \ldots \cup P_p \cup Q_1 \cup \ldots \cup Q_q$ has no directed circuits.

A $(p,q)$-web $(P_1, \ldots, P_p, Q_1, \ldots, Q_q)$ is a *$(p,q)$-mesh* if for $1 \leq i \leq p$ and for all $j, j'$ with $1 \leq j < j' \leq q$, if $v \in V(P_i \cap Q_j)$ and $v' \in V(P_i \cap Q_{j'})$ then $v$ occurs before $v'$ in $P_i$. The first main result of this section is the following.

**(4.1)** *For every integer $p \geq 1$ there is an integer $p' \geq 1$ such that every digraph with a $(p', p')$-web has a $(p,p)$-mesh.*

To prove (4.1) we need a more complicated object, as follows. A $(p,q)$-web $(P_1,\dots,P_p,Q_1,\dots,Q_q)$ is called a *$(p,q,r,s)$-sheaf* if

(i) $r,\ s\geq 0$ are integers and $r+s\leq q$,

(ii) for $1\leq j\leq r$ and for $1\leq i,\ i'\leq p$ with $i\neq i'$, if $v_1,v_2\in V(P_i\cap Q_j)$ and $v'\in V(P_{i'}\cap Q_j)$ then $v'$ does not lie in $Q_j$ between $v_1$ and $v_2$,

(iii) for $r+1\leq j<j'\leq r+s$ and for $1\leq i\leq p$, if $v\in V(P_i\cap Q_j)$ and $v'\in V(P_i\cap Q_{j'})$ then $v$ occurs before $v'$ in $P_i$.

We shall need the following four simple observations.

**(4.2)** (i) *If $G$ has a $(p,q)$-web then it has a $(p,q,0,0)$-sheaf.*

(ii) *If $G$ has a $(p,q,r,0)$-sheaf with $r<q$ then it has a $(p,q,r,1)$-sheaf.*

(iii) *If $G$ has a $(p,q,p!p,0)$-sheaf then it has a $(p,p)$-mesh.*

(iv) *If $G$ has a $(p,q,0,p)$-sheaf then it has a $(p,p)$-mesh.*

**Proof.** Every $(p,q)$-web is a $(p,q,0,0)$-sheaf, so (i) follows. If $r<q$, every $(p,q,r,0)$-sheaf is a $(p,q,r,1)$-sheaf, so (ii) follows. For (iii), let $(P_1,\dots,P_p,Q_1,\dots,Q_q)$ be a $(p,q,p!p,0)$-sheaf. For $1\leq j\leq p!p$ there is a permutation $\pi$ of $\{1,\dots,p\}$ so that for $1\leq i<i'\leq p$, every vertex of $P_{\pi(i)}\cap Q_j$ occurs in $Q_j$ before every vertex of $P_{\pi(i')}\cap Q_j$. Since there are only $p!$ such permutations, there are at least $p$ values of $j$ that yield the same permutation $\pi$, say $j_1,\dots,j_p$. Then

$$(Q_{j_1},\dots,Q_{j_p},P_{\pi(1)},\dots,P_{\pi(p)})$$

is a $(p,p)$-mesh. This proves (iii). For (iv), let $(P_1,\dots,P_p,Q_1,\dots,Q_q)$ be a $(p,q,0,p)$-sheaf. Then $(P_1,\dots,P_p,Q_1,\dots,Q_p)$ is a $(p,p)$-mesh. ∎

The following is somewhat less trivial.

**(4.3)** *If $G$ has a $(p^2,q,r,s)$-sheaf where $s\geq 1$ and $r+s<q$, then it has either a $(p,q,r,s+1)$-sheaf or a $(p,q,r+1,s-1)$-sheaf.*

**Proof.** Let $(P_1,\dots,P_{p^2},Q_1,\dots,Q_q)$ be a $(p^2,q,r,s)$-sheaf. For $1\leq i\leq p^2$ let $F_i$ be the minimal path of $Q_{r+s}$ that includes $V(P_i\cap Q_{r+s})$.

Suppose first that some edge $e$ of $Q_{r+s}$ belongs to $F_i$ for at least $p$ values of $i$, say $i_1,\dots,i_p$. Let $Q$, $Q'$ be the two paths obtained from $Q_{r+s}$ by deleting $e$, where $Q$ contains the tail of $e$. For $i=i_1,\dots,i_p$, $P_i$ meets both $Q$ and $Q'$, and every vertex in $P_i\cap Q$ occurs in $P_i$ before every vertex of $P_i\cap Q'$ since $P_1\cup\dots\cup P_{p^2}\cup Q_1\cup\dots\cup Q_q$ has no directed circuits. Hence

$$(P_{i_1},\dots,P_{i_p},Q_1,\dots,Q_{r+s-1},Q,Q',Q_{r+s+1},\dots,Q_{q-1})$$

is a $(p,q,r,s+1)$-sheaf, as required.

We may assume then that every edge of $Q_{r+s}$ occurs in $F_i$ for fewer than $p$ values of $i$. Consequently, there are $p$ values of $i$, say $i_1,\dots,i_p$, so that either

$F_{i_1},\dots,F_{i_p}$ each have only one vertex, or $F_{i_1},\dots,F_{i_p}$ each have at least two vertices and are mutually edge-disjoint. Then in either case, $F_{i_1},\dots,F_{i_p}$ are pairwise vertex-disjoint, and

$$(P_{i_1},\dots,P_{i_p},Q_1,\dots,Q_r,Q_{r+s},Q_{r+1},Q_{r+2},\dots,Q_{r+s-1},Q_{r+s+1},\dots,Q_q)$$

is a $(p,q,r+1,s-1)$-sheaf. ∎

We deduce the following, which immediately implies (4.1).

**(4.4)** *Let $p\geq 1$ be an integer, and let $q=2p!p+p$. If $G$ has a $(p^{2^q},q)$-web then it has a $(p,p)$-mesh.*

**Proof.** Let $G$ have a $(p^{2^q},q)$-web. By (4.2)(i), it has a $(p^{2^q},q,0,0)$-sheaf. Choose $r,\ s\geq 0$ and $n=2r+s$, so that $n\leq q$ and $G$ has a $(p^{2^{q-n}},q,r,s)$-sheaf, with $n$ maximum. If $r\geq p!p$ then $G$ has a $(p,q,p!p,0)$-sheaf and hence a $(p,p)$-mesh by (4.2)(iii). If $s\geq p$ then $G$ has a $(p,q,0,p)$-sheaf and hence a $(p,p)$-mesh by (4.2)(iv). We may suppose then (for a contradiction) that $s<p$ and $r<p!p$, and so $n<q$. By (4.2)(ii), $s\neq 0$; and so by (4.3) $G$ has either a $(p^{2^{q-n-1}},q,r,s+1)$-sheaf or a $(p^{2^{q-n-1}},q,r+1,s-1)$-sheaf. In either case this contradicts the maximality of $n$. The result follows. ∎

A $(p,q)$-web $(P_1,\dots,P_p,Q_1,\dots,Q_q)$ is a $(p,q)$-*grid* if

(i) for $1\leq i\leq p$ and $1\leq j\leq q$, $P_i\cap Q_j$ is a path $R_{ij}$ say

(ii) for $1\leq i\leq p$ the paths $R_{i1},\dots,R_{iq}$ are in order in $P_i$, and

(iii) for $1\leq j\leq q$ the paths $R_{1j},\dots,R_{pj}$ are in order in $Q_j$.

Our next main result is the following.

**(4.5)** *For every integer $p\geq 1$ there is an integer $p'\geq 1$ such that every digraph with a $(p',p')$-mesh has a $(p,p)$-grid.*

To prove (4.5) we need the following lemma.

**(4.6)** *Let $r\geq 1$ be an integer, and let $p=\frac{1}{2}r(r+1)$. Let $G$ have no directed circuit, and let $P_1,\dots,P_p$ be disjoint paths in $G$. Let $Q$ be a path in $G$ meeting all of $P_1,\dots,P_p$. There is a path $R$ in $G$ so that $R\cap P_i$ is a path for at least $r$ values of $i$ $(1\leq i\leq p)$.*

**Proof.** Choose a path $Q$ meeting all of $P_1,\dots,P_p$, so that $Q\cap(P_1\cup\dots\cup P_p)$ has as few components as possible. Let the components of $Q\cap(P_1\cup\dots\cup P_p)$ be $C_1,\dots,C_n$, numbered in order on $Q$. For $1\leq i\leq n$, let $f(C_i)=j$ where $C_i\subseteq P_j$. Suppose that $Q\cap P_i$ is a path for $<r$ values of $i$. Hence for all other values of $i$, $Q\cap P_i$ has at least two components, and so

$$n>r+2(p-r)=r^2.$$

Let us say $C_i$ is *good* if $Q \cap P_j = C_i$, where $j = f(C_i)$. There are $< r$ good components, and so there exists $i$ with $1 \le i \le n-r+1$ so that $C_i, C_{i+1}, \ldots, C_{i+r-1}$, are all not good. If $f(C_i), f(C_{i+1}), \ldots, f(C_{i+r-1})$ are all distinct then we may take $R$ to be the minimal subpath of $Q$ containing $C_i, C_{i+1}, \ldots, C_{i+r-1}$. Thus we may assume that there exist $j$, $k$ with $i \le j < k \le i+r-1$ so that $f(C_j) = f(C_k) = h$ say.

Choose such $j, k$ with $k-j$ minimum. Let $Q'$ be obtained from $Q$ by replacing the path of $Q$ between $C_j$ and $C_k$ by the path $P_h$ between $C_j$ and $C_k$ (this exists and has no vertex in $Q$ except its ends, since $G$ has no directed circuits). Since none of $C_{j+1}, \ldots, C_{k-1}$ are good, and $f(C_{j+1}), \ldots, f(C_{k-1})$ are all distinct by the minimality of $k-j$, it follows that $Q'$ meets all of $P_1, \ldots, P_p$, contradicting the minimality of $n$.

Thus, if $Q \cap P_i$ is a path for $< r$ values of $i$ then the result holds. On the other hand, if it is a path for $\ge r$ values of $i$ then we may take $R = Q$, and again the result holds. ∎

**Proof of (4.5).** Let $p' = p(\frac{1}{2}p(p+1))^p$. We claim that $p'$ satisfies (4.5). For let $G$ have a $(p', p')$-mesh $(P_1, \ldots, P_{p'}, Q_1, \ldots, Q_{p'})$. For $1 \le i, j \le p'$ let $P_{ij}$ be the minimal subpath of $P_i$ that contains $P_i \cap Q_j$. Then $P_{i1}, \ldots, P_{ip'}$ are vertex-disjoint paths of $P_i$, in order in $P_i$.

Let $1 \le j \le p'$, and let $G_j = Q_j \cup P_{1j} \cup P_{2j} \cup \ldots \cup P_{mj}$ where $m = \frac{1}{2}p(p+1)$. Then $P_{1j}, \ldots, P_{mj}$ are disjoint paths of $G_j$, and $Q_j$ meets them all. By (4.6) there is a path $R_j$ of $G_j$ so that $R_j \cap P_{ij}$ is a path for $\ge p$ values of $i$ $(1 \le i \le m)$. Choose a sequence $X_j$ of $p$ distinct members of $\{1, \ldots, m\}$, say $i_1, \ldots, i_p$, so that for $1 \le k \le p$, $R_j \cap P_{i_k j}$ is a path and hence $R_j \cap P_{i_k}$ is a path, and moreover $R_j$ meets $P_{i_1}, \ldots, P_{i_p}$ in order. Since there are $\le m^p = (\frac{1}{2}p(p+1))^p = p'/p$ possibilities for $X_j$, there are at least $p$ values of $j$ for which $X_j$ is the same; let these values be $j_1, \ldots, j_p$ say, where $j_1 < \ldots < j_p$. Let the common value of $X_j$ be $\{i_1, \ldots, i_p\}$ say, where $i_1 < \ldots < i_p$. Then

$$(P_{i_1}, \ldots, P_{i_p}, R_{j_1}, \ldots, R_{j_p})$$

is a $(p,p)$-grid. ∎

Next, we need:

**(4.7)** *For all integers $p, q \ge 1$ there is an integer $p' \ge 1$ so that for every digraph $G$, if $G$ contains a $(p', p')$-grid then it contains a $(p,q)$-fence.*

**Proof.** Let $p' = pq+1$, and let $(P_1, \ldots, P_{p'}, Q_1, \ldots, Q_{p'})$ be a $(p', p')$-grid in $G$. For $1 \le i \le p'$ and $1 \le j \le p'$ let $T_{ij} = P_i \cap Q_j$, and let $T_{ij}$ have first vertex $a(i,j)$ and last vertex $b(i,j)$. For $1 \le i \le p$, let $i' = q(i-1)+1, j' = qi, j'' = j'+1$, and let

$$R_{2i-1} = P_{i'}[a(i', i'), b(i', j')]$$
$$R_{2i} = Q_{j''}[a(i', j''), b(j', j'')].$$

For $j=1,\dots,q$ let $S_j$ be the union of all the paths

$$Q_j[a(1,j),b(q-j+1,j)]$$
$$Q_{tq+j}[a(tq-j+1,tq+j),b((t+1)q-j+1,tq+j)] \quad (1\leq t\leq p-1)$$
$$P_{tq-j+1}[a(tq-j+1,(t-1)q+j),b(tq-j+1,tq+j)] \quad (1\leq t\leq p-1)$$
$$P_{pq-j+1}[a(pq-j+1,(p-1)q+j),b(pq-j+1,pq+1)].$$

It is routine to check that $(R_1,\dots,R_{2p},S_1,\dots,S_q)$ is a $(p,q)$-fence. ∎

**Proof of (2.3).** Let $n\geq 1$ be an integer. Let $p_1,r_1\geq 1$ satisfy (3.1) (with $p,r$ replaced by $p_1,r_1$). Let $q_1=2r_1$. Let $p_2$ satisfy (4.7) (with $p,q,p'$ replaced by $p_1,q_1,p_2$). Let $p_3$ satisfy (4.5) (with $p,p'$ replaced by $p_2,p_3$). Let $p_4$ satisfy (4.1) (with $p,p'$ replaced by $p_3,p_4$). We claim that $k=p_4$ satisfies (2.3). For the hypothesis of (2.3) implies that $G$ has a $(p_4,p_4)$-web. By (4.1) it has a $(p_3,p_3)$-mesh. By (4.5) it has a $(p_2,p_2)$-grid. By (4.7) it has a $(p_1,q_1)$-fence. Let the top and bottom of the fence be $A$ and $B$. There are $q_1=2r_1$ disjoint paths in $G$ from $A$ to $B$. Let $(X,Y)$ be a separation of $G$ with $B\subseteq X$ and $A\subseteq Y$. Let $C=B\cup(X-Y)$, $D=V(G)-C$. Thus $A\subseteq D$. Since there are $2r_1$ disjoint paths from $A$ to $B$, there are at least $2r_1$ edges of $G$ with tail in $D$ and head in $C$. Since every vertex has indegree equal to outdegree, there are also $\geq 2r_1$ edges with head in $D$ and tail in $C$. But for every such edge (with tail $c$ and head $d$ say) it is not the case that $c\in X-Y$ and $d\in Y-X$, since $(X,Y)$ is a separation; and so either $c\in B\cap X\cap Y$, or $d\in X\cap Y-B$. There are at most $2|B\cap X\cap Y|$ edges with tail in $B\cap X\cap Y$, since $G$ is divalent, and similarly there are at most $2|X\cap Y-B|$ with head in $X\cap Y-B$. Hence

$$2r_1\leq 2|B\cap X\cap Y|+2|X\cap Y-B|=2|X\cap Y|,$$

and so $|X\cap Y|\geq r_1$. From (1.4) it follows that there are $r_1$ disjoint paths in $G$ from $B$ to $A$. From (3.1), $\nu(G)\geq n$ as required. ∎

## 5. An algorithm

In this section we use (1.3) to obtain a polynomial-time algorithm to test if a digraph $G$ has $\nu(G)\geq n$, for fixed $n$. Indeed, we can solve a more general problem, as follows.

Let $G$, $H$ be digraphs. An *embedding of $H$ in $G$* is a function $\phi$ with domain $V(H)\cup E(H)$, such that

(i) $\phi(v)\in V(G)$ for $v\in V(H)$, and $\phi(v_1)\neq\phi(v_2)$ for distinct $v_1,v_2\in V(H)$

(ii) for every edge $e$ of $H$ with tail $u$ and head $v$, if $u\neq v$ then $\phi(e)$ is a directed path in $G$ from $\phi(u)$ to $\phi(v)$, and if $u=v$ then $\phi(e)$ is a directed circuit with $\phi(u)\in V(\phi(e))$

(iii) for all $e \in E(H)$ and $v \in V(H)$, if $e$ is not incident with $v$ then $\phi(v) \notin V(\phi(e))$

(iv) for all distinct $e_1, e_2 \in E(H)$, $E(\phi(e_1) \cap \phi(e_2)) = \emptyset$, and for every vertex $w \in V(\phi(e_1) \cap \phi(e_2))$ there exists $v \in V(H)$ with $\phi(v) = w$.

In other words, there is an embedding of $H$ in $G$ if and only if there is a subdigraph of $G$ that is isomorphic to a "directed subdivision" of $H$.

Let $\psi$ be an injective function from a subset $dom(\psi)$ of $V(H)$ into $V(G)$. An embedding $\phi$ of $H$ in $G$ *extends* $\psi$ if $\phi(v) = \psi(v)$ for all $v \in dom(\psi)$.

The main result of this section is the following.

**(5.1)** *Let $n$ be a fixed integer, and $H$ a fixed digraph. There is an algorithm, with running time bounded by a polynomial in $|V(G)| + |E(G)|$, which has*

(a) *input a digraph $G$ and an injective function $\psi$ as above, and*

(b) *output one of the following:*

  (i) *$\psi$ can be extended to an embedding of $H$ in $G$*

  (ii) *$\psi$ cannot be so extended*

  (iii) *$\nu(G) \geq n$.*

If we omit (iii) there is unlikely to be any such algorithm, because the problem was shown to be NP-complete by Fortune, Hopcroft and Wyllie [3], even when $H$ has just two components, each a non-loop edge.

As a corollary we have

**(5.2)** *For every fixed integer $n$, there is a polynomial-time algorithm to decide whether $\nu(G) \geq n$ for a given input digraph $G$.*

**Proof.** We apply (5.1), taking $H$ to be the digraph with $n$ components each with one vertex and one (loop) edge, and taking $dom(\psi) = \emptyset$. ∎

To prove (5.1) we need the following.

**(5.3)** *Let $H$ be a fixed digraph. There is an algorithm, with running time polynomial in $|V(G)| + |E(G)|$, which has*

(i) *input a digraph $G$ and an injective function $\psi$ from $V(H)$ into $V(G)$, so that every directed circuit of $G$ meets $\{\psi(v) : v \in V(H)\}$*

(ii) *output whether $\psi$ can be extended to an embedding of $H$ in $G$.*

**Proof.** (Sketch.) "Split" each vertex of $G$ in $\{\psi(v) : v \in V(H)\}$ into two vertices, one the head of all edges of $G$ for which $\psi(v)$ was the head, and the other the tail of all edges of $G$ for which $\psi(v)$ was the tail. Let the new digraph we obtain be $G'$. Similarly, split each vertex of $H$, forming $H'$, and define $\psi' : V(H') \to V(G')$ in the natural way. Now $\psi$ extends to an embedding of $H$ in $G$ if and only if $\psi'$ extends to an embedding of $H'$ in $G'$; and since $G'$ has no directed circuits, the latter question can be answered by applying an algorithm of Fortune, Hopcroft and Wyllie [3].

**Proof of (5.1).** (Sketch.) First we find $X \subseteq V(G)$ meeting all directed circuits of $G$, with $|X| \le t_n$. (If there is no such $X$ then $\nu(G) \ge n$ by (1.3) and we are done.) If $v \in V(H) - dom(\psi)$, then there are only $|V(G)|$ possibilities for $\phi(v)$, and therefore we can enumerate them all and test them separately. Consequently, we can assume that $dom(\psi) = V(H)$. If $x \in X$ and $x \notin \{\psi(v) : v \in V(H)\}$, then either $x$ is an internal vertex of exactly one path $\phi(e)$ ($e \in E(H)$) or of none. We can examine the first case by subdividing $e$ in $H$ with a new vertex $v$ say, and defining $\psi(v) = x$, and we can examine the second case by adding a new vertex $v$ to $H$ of degree 0, and defining $\psi(v) = x$. By repeating this for all such $x$, we may therefore assume that $X \subseteq \{\psi(v) : v \in V(H)\}$. But then the result follows from (5.3). ∎

It is easy to modify the algorithm of (5.2) to *find* the $n$ disjoint circuits if they exist. Roughly, we delete the vertices of $G$ one at a time, and apply the algorithm of (5.2) at each step, until we have deleted so much that we find the circuits no longer exist. Replace the last vertex deleted. Now the circuits do exist, and also there is a bounded size set of vertices meeting all directed circuits, so we can apply (5.3) as before (using a modification of (5.3) that finds the desired embedding of $H$).

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